Talk 7 p-adic period rings

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Recall: $C \supseteq \mathbb{Q}_p$ complete, alg closed. $\underline{Sym}$. cat of sympathetic algs.

$S = \mathbb{Q}_p$-vector spaces or $\mathbb{Q}_p$-Banach spaces, rings, top rings.

$\mathbb{T}: Sym \to S$. satisfying.

T1). $\mathrm{Spec}(\Lambda) \times \mathbb{T}(\Lambda) \to \mathbb{T}(C)$. is continuous.
$(s, \lambda) \mapsto \mathbb{T}(s)(x)$.

T2). $\mathbb{T}(\Lambda) \to \mathrm{Hom}_{cont}(\mathrm{Spec}(\Lambda), \mathbb{T}(C))$ is injective.

Def. $S = \mathbb{Q}_p$-v.s., $\mathbb{T}$ is called $\underline{\text{Vector Space}}$. rings, Rings.
$S = \mathbb{Q}_p$-B.s. $\mathbb{T}$ ______ $\underline{\text{Banach Space}}$ top rings, Top Ring.

e.g. 1). $V \in \mathbb{Q}_p$-v.s. (resp. $\mathbb{Q}_p$-B.s.).
$\underline{V}: \Lambda \mapsto V$. is a V.S. (resp. B.S.).

2). $\mathbb{V}^r(\Lambda) = \Lambda^d$. is a B.S.

3) $\mathbb{A}(\Lambda) = \mathcal{O}_\Lambda$, $\mathbb{B}(\Lambda) = \Lambda$.
$\left(\|\lambda\|_\Lambda = \sup_{s \in \mathrm{Spec}(\Lambda)} |s(\lambda)| \Rightarrow T_2 \right)$.

Def. A B.S. is finite dimensional. (f.d.) if it is f.d. V.S.

$\phi: \mathbb{W}_1 \to \mathbb{W}_2$. a morphism of B.S., then $\ker(\phi)$ is B.S.

- $\phi(\Lambda)$ is continuous, $\Rightarrow \ker(\phi(\Lambda)) \subseteq \mathbb{W}_1(\Lambda)$ is closed and is a B.s.
- But $\mathrm{im}\,\phi(\Lambda)$ may not be closed in $\mathbb{W}_2(\Lambda)$.

Prop. If ϕ is a morphism of f.d. B.S, then $\mathrm{Im}\,\phi$ is a f.d. B.S.

K. CDVF. mixed char $(0,p)$, k residue field, perfect. π uniformizer.
$K_0 = W(k)[\frac{1}{p}]$, $e = [K:K_0]$, $P \in K_0[x]$. minimal poly. of π.

Ring. $\mathbb{A}, \mathbb{B}$, $\Lambda \mapsto \mathcal{O}_\Lambda, \Lambda$. resp.

$\varprojlim_{\substack{n \in \mathbb{N} \\ x \mapsto x^p}} \mathbb{A}/\pi\mathbb{A} := \mathbb{R}$. Ring. of char p. (may replace π by any $a \in \mathfrak{m}_C - p\mathfrak{m}_C$).

$\Lambda \in \underline{Sym}$. $\mathbb{R}(\Lambda) \ni x = (x_n)_{n \geq 0}$. $x_n \in \mathbb{A}(\Lambda)/\pi\mathbb{A}(\Lambda)$, $x_{n+1}^p = x_n$ $\forall n$.

$R = \mathbb{R}(C)$. (bold. $\mathbb{A}, \mathbb{R}, \mathbb{B}, \mathbb{I}$ ... for Ring ...., A, R, B, I. denotes valuation at C.)

$k_C \hookrightarrow R.$

$x \xmapsto{x_n} ([x^{p^{-n}}])_{n\geq 0}$, $x_n = $ Im. of $[x^{p^{-n}}]$ via $W(k_C) \to \mathcal{O}_C \to \mathcal{O}_C/\pi$.

$\forall \Lambda \in \underline{Sym}$, $\mathbb{R}(\Lambda)$ is perfect R-alg of char p.

$\mathbb{R}(\Lambda) \longrightarrow \prod_{n\geq 0} \mathcal{O}_\Lambda$ is multiplicative.

$x = (x_n) \quad (x^{(m)})_{m\geq 0}$. $\quad x^{(m)} = \lim_n (\hat{x}_{n+m})^{p^n}$ and is independent of liftings.

Def: $\|\cdot\|_{\mathbb{R}}$ norm on $\mathbb{R}(\Lambda)$ by $\|x\|_{\mathbb{R}} = \|x^{(0)}\|_\Lambda$

For $x, y \in \mathbb{R}(\Lambda)$, $\|x-y\|_{\mathbb{R}} \leq p^{-1} \Leftrightarrow x_0 - y_0 = 0$ in $\mathcal{O}_\Lambda/p\mathcal{O}_\Lambda \Leftrightarrow \|x^{(0)} - y^{(0)}\|_\Lambda \leq p^{-1}$.

if $\|x^{(n)} - y^{(n)}\|_\Lambda \leq p^{-1} \Rightarrow \|x-y\|_{\mathbb{R}} = \underbrace{\|x^{p^{-n}} - y^{p^{-n}}\|}_{\substack{\shortparallel \\ x-y+p^n(\text{—})}}{}^{p^n}_{\mathbb{R}} \leq p^{-p^n}$. $(*)$.

Prop. $\mathbb{R}$ is a Top. Ring.

pf. Follows from $\mathbb{A}$ is a Top Ring.

T1: fix s_0, λ_0. $\quad (|s(a)| \leq \|a\|_\Lambda)$. $\Downarrow$

If $s \in \mathrm{Spec}(\Lambda)$, $\lambda \in \mathbb{R}(\Lambda)$, then $\|s(\lambda)\|_{\mathbb{R}} \leq \|\lambda\|_{\mathbb{R}}$.

then $\|s(x) - s_0(\lambda_0)\|_{\mathbb{R}} \leq \sup(\|\lambda - \lambda_0\|_{\mathbb{R}}, \|s(\lambda_0) - s_0(\lambda_0)\|_{\mathbb{R}})$.

Recall: opens $U(n, \lambda, x, \varepsilon) = \{s \in \mathrm{Spec}(\Lambda), |s(\lambda_i) - x_i| < \varepsilon_i \ \forall 1 \leq i \leq n\}$ form a basis.
$\lambda_i \in \Lambda$, $x_i \in C$.

$\forall n \geq 1$, $\exists$ open $U_n \subseteq \mathrm{Spec}(\Lambda)$ s.t. if $s \in U_n$, then $|s(\lambda_0^{(n)}) - s_0(\lambda_0^{(n)})|_C \leq p^{-1}$.

$\Rightarrow$ $\begin{array}{l} \mathrm{Spec}(\Lambda) \times \mathbb{R}(\Lambda) \to R \\ (s, \lambda) \mapsto s(\lambda) \end{array}$ is continuous. $\quad \xRightarrow{(*)} \|s(x_0) - s_0(\lambda_0)\|_{\mathbb{R}} \leq p^{-p^n}$.

T2: $\|x\|_{\mathbb{R}} = \sup_{s \in \mathrm{Spec}(\Lambda)} \|s(x)\|_{\mathbb{R}}$. $\quad (\|\lambda\|_\Lambda := \sup_s |s(\lambda)|)$.

$\Longrightarrow \mathbb{R}(\Lambda) \times \mathrm{Spec}(\Lambda) \to R$ is injective.

$\mathbb{A}_{\inf, K}$. $\quad \mathbb{A}_{\inf} = W(\mathbb{R})$. p-torsion free. $x = \sum_{n\geq 0} p^n [x_n]$.

$\varphi: \mathbb{A}_{\inf} \to \mathbb{A}_{\inf}$. $\quad [x_n] \mapsto [x_n^p]$.

$\mathbb{A}_{\inf, K} = \mathbb{A}_{\inf} \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_K$.

$\theta: \mathbb{A}_{\inf, K}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$. $\quad \sum \pi^n [x_n] \mapsto \sum \pi^n x_n^{(0)}$.

$\mathbb{J}_K = \mathrm{Ker}(\theta)$. $\quad \mathbb{I}_K = \mathrm{Ker}(\mathbb{A}_{\inf, K} \xrightarrow{\theta} \mathbb{A} \to \mathbb{A}/\pi\mathbb{A})$.

Prop. 1). $\mathbb{R} \longrightarrow \mathbb{A}/\pi\mathbb{A}$. $\quad x \mapsto x_0$ is surjective and Ker. is principal gen by $\bar{\alpha} \in R$ s.t. $\|\bar{\alpha}\|_{\mathbb{R}} = |\pi|$.

2). θ is surjective, $\mathbb{J}_K$ is principal, $\alpha \in \mathbb{J}_K$ is a gen iff $\|\bar{\alpha}\|_{\mathbb{R}} = |\pi|$.

Lem: For $x = (x^{(m)}) \in \mathbb{R}(\Lambda)$, $\alpha = (\alpha^{(m)}) \in R$, $\|\alpha\|_{\mathbb{R}} \geq \|x\|_{\mathbb{R}}$, then $\exists!$ $y \in \mathbb{R}(\Lambda)$ s.t. $x = \alpha \cdot y$.

pf. $\|x^{(0)}\| \leq \|\alpha^{(0)}\| \xRightarrow{\text{Spectral}} \|x^{(m)}\| \leq \|\alpha^{(m)}\|$, then set $y^{(m)} = x^{(m)}/\alpha^{(m)} \in \mathcal{O}_\Lambda$ □.

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2). Surjectivity follows from 1) by reduction mod π.

Generator, for α as above, if: $x = [x_0] + \pi [x_1] + \cdots \in \mathbb{J}_K(\Lambda)$, need to find $y \in \mathbb{A}_{\inf,K}(\Lambda)$ s.t. $x - \alpha y \in \pi \mathbb{A}_{\inf,K}$.

$\theta(x) = 0 \Rightarrow \|x_0\|_R \geq \|\bar{\alpha}\|_R$. $\exists\ \bar{y} \in R$ s.t. $x_0 = \bar{\alpha} \cdot \bar{y}$, take $y = [\bar{y}]$. $\square$

Topology: $\mathbb{A}_{\inf,K}(\Lambda)$, $\mathbb{I}_K(\Lambda)$-adic top. $\mathbb{I}_K(\Lambda)$ is gen by π, $\bar{\omega} = [\underline{\pi}] - \pi$, where $\underline{\pi} \in R$, $\underline{\pi}^{(0)} = \pi$.

$\mathbb{I}^{n,k}_K(\Lambda)$ gen by $\pi^n, \bar{\omega}^k$ (Colmez, use p instead of π)

Prop. ($\mathbb{A}_{\inf}$.1) 1) $(x_n)_{n\geq 0} \longmapsto \sum_{n\geq 0} \pi^n [x_n]$ defines a homeomorphism $R^{\mathbb{N}} \xrightarrow{\sim} \mathbb{A}_{\inf,K}$. $(*)$

$\Downarrow$ 2). $\mathbb{A}_{\inf,K}$ is a Top Ring.

Lem 1) $\mathbb{A}_{\inf,K} \to R$ $x \longmapsto \bar{x}$ reduction is continuous, (easy).

2). $R \to \mathbb{A}_{\inf,K}$. $x \longmapsto [x]$ is continuous.

pf. $\underline{[x+y] - [x] = \sum_{n\geq 0} p^n}\, [Q_n(x^{p^{-n}}, y^{p^{-n}})]$ where $Q_n \in \mathbb{Z}[X,Y]$, $Y \mid Q_n(X,Y)$

If $\|y\|_R \leq |p|^{p^k}$, then $[x+y] - [x] \in \mathbb{I}^k_K$ $\Rightarrow$ continuity. $\square$

pf. Lem 2) $\Rightarrow$ $(*)$ is continuous as a uniform limit of continuous maps.

On the other hand, we can recover $[x_n]$ from by "reduction mod π". $\square$

($x \in \mathbb{A}_{\inf,K}$, $a_0 = x$, $x_0 = \bar{a}_0$, $x_n = \bar{a}_n$, $a_{n+1} = \frac{1}{\pi}(a_n - [x_n])$ ...).

Prop ($\mathbb{A}_{\inf}$.2) TFAE. 1) $x \in \mathbb{I}^{n,k}_K(\Lambda)$. 2). $\forall s \in \mathrm{Spec}(\Lambda)$, $s(x) \in \mathbb{I}^{n,k}_K$.

pf. 1) $\Rightarrow$ 2) is clear. 2) $\Rightarrow$ 1) induction on k.

$k=1$. (ii) $\Leftrightarrow$ $s(\theta(x)) \in \pi^n \mathcal{O}_C$ $\forall s \in \mathrm{Spec}(\Lambda)$. $\Rightarrow$ $\theta(x) \in \pi^n \mathcal{O}_\Lambda$.

Since θ is surj, $\exists\ a \in \mathbb{A}_{\inf,K}$ s.t. $\theta(a) = \pi^{-n} \theta(x)$, so $x = \pi^n a + b$. $b \in \ker(\theta)$. $\Rightarrow x \in \mathbb{I}^{n,1}$.

From $k \to k+1$, if $s(x) \in \mathbb{I}^{n,k+1}_K$ $x = \pi^n a + \bar{\omega} b$ as above.

then. $s(b) \in \mathbb{I}^{n,k}_{n,k}$ (E.x). $\xRightarrow{\text{induction}}$ $b \in \mathbb{I}^{n,k}(\Lambda)$. $\square$

($x = s(x)$ ....., $x = \pi^n b + \bar{\omega}^{k+1} c$. and $\theta(b) = \theta(a)$. $\Rightarrow b = a + \bar{\omega} \cdot y$. $\Rightarrow \frac{x - \pi^n a}{\bar{\omega}} = \pi^n y + \bar{\omega}^k c$).
for some $a \in \mathbb{A}_{\inf,K}$.

$\mathbb{B}^+_{dR}$. $\theta: \mathbb{A}_{\inf}[\frac{1}{p}]$ (resp. $\mathbb{A}_{\inf,K}[\frac{1}{p}]$) $\to \mathbb{B}$ $\mathbb{J} := \ker\theta = [\underline{p}] - p$, ($\mathbb{J}_K = [\underline{\pi}] - \pi$)

$\mathbb{B}_m = \mathbb{A}_{\inf}[\frac{1}{p}] / \mathbb{J}^m$. $\mathbb{B}^+_{dR} := \varprojlim \mathbb{B}_m$. ($\mathbb{B}_{m,K}$, $\mathbb{B}^+_{dR,K}$).

Topology: $\forall \Lambda \in \mathrm{Sym}$, $\mathbb{B}_m(\Lambda)$. $\|\cdot\|_m$ def by
$\|x\|_m = 1$ iff $x \in \mathrm{Im}(\mathbb{A}_{\inf}) - \mathrm{Im}(p\mathbb{A}_{\inf})$, $\|p^k x\|_m = p^{-k}$. (similar for $\mathbb{B}_{m,K}$).

(top. $\mathbb{Q}_p$-v.s. on $\mathbb{B}_m(\Lambda)$).

Prop ($\mathbb{A}_{\inf}$.3) $\Rightarrow$ ... [illegible] (top. $\mathbb{Q}_p$-v.s. on $\mathbb{B}_m(\Lambda)$).

Prop(A_{inf},2). $\Rightarrow$ $\|\lambda\|_m = \sup_{s\in Spec(\Lambda)} \|s(\lambda)\|_m$.

$\mathbb{B}^+_{dR}(\Lambda)$. proj limit top.

A continous K-linear section. $s: \mathcal{O}_\Lambda \to \mathbb{A}_{inf,K}(\Lambda)$ of θ.

Let $\{e_i\}_{i\in I} \subseteq \mathcal{O}_\Lambda$ s.t. images in $\mathcal{O}_\Lambda/\pi\mathcal{O}_\Lambda$ is a basis $/k$.

$\forall x\in\Lambda$, $x=\sum x_i e_i$, $x_i\in K$.

Take $\tilde{e}_i \in \mathbb{A}_{inf,K}(\Lambda) \xrightarrow{\theta} e_i$, then $s(\sum x_i e_i) = \sum_i x_i \tilde{e}_i$ is a section.

Take $v \in J_K$ a generator, $\rightsquigarrow$ $\tilde{\theta}_v$ $\mathbb{B}^+_{dR,K}(\Lambda) \xrightarrow{\sim} \Lambda[[X]]$ of top K-v.s.

(not a ring homomorphism).

$\forall x\in \mathbb{B}^+_{dR,K}(\Lambda)$ $\rightsquigarrow$ define $\{a_n(x)\}$, $\{b_n(x)\}$ by

1) $a_0(x)=x$. 2) $b_n(x) = \theta(a_n(x))$. and $a_{n+1} = \frac{1}{v}(a_n(x) - s(b_n(x)))$.

$\rightsquigarrow$ $x \longmapsto \tilde{\theta}_v(x) = \sum_{n\geq 0} b_n(x) X^n$. ($v \longmapsto X$)

satisfying. $\forall F\in \mathbb{Q}_p[[X]]$. $\tilde{\theta}_v(x\cdot F(v)) = \tilde{\theta}_v(x)\cdot F$.

Prop. $\mathbb{B}_{m,K}$, $\mathbb{B}^+_{dR,K}$ are Top. Rings, $\mathbb{B}_{m,K}$, Banach Ring.

pf. T2). $\mathbb{B}_{m,K}(\Lambda) \to Hom(Spec(\Lambda), B_{m,K})$ is injective. follows from

$\|\lambda\|_m = \sup_s \|\lambda(s)\|$. Prop($A_{inf}$2)

T1). $\mathbb{B}_{m,K}(\Lambda)\times Spec(\Lambda) \to B_{m,K}$ is continous.

$(\lambda, s) \longmapsto s(\lambda)$.

Given $\lambda_0 = \pi^{-r}\mu_0 \in \mathbb{A}_{inf,K}[\pi^{-1}]$, and $s_0\in Spec(\Lambda)$.

By Prop(A_{inf},1). $\forall n, \exists U$ nbh of s in $Spec(\Lambda)$ s.t.

$s(\mu_0) - s_0(\mu_0) \in \mathbb{I}^{n+r,m}_K(\Lambda)$.

If $\lambda - \lambda_0 \in \pi^n \mathbb{A}_{inf,K}(\Lambda) + J^m_K(\Lambda)$, $s\in U \Rightarrow$

$s(\lambda) - s_0(\lambda_0) \in \pi^n \mathbb{A}_{inf,K}(\Lambda) + \mathbb{J}^m_K(\Lambda)$. $\square$

Prop. $\mathbb{B}_m \to \mathbb{B}_{m,K}$, $\mathbb{B}^+_{dR} \to \mathbb{B}^+_{dR,K}$ are iso,

pf. show it for $\mathbb{B}_m$. it suffices to show $\mathbb{B}_m(\Lambda) \to \mathbb{B}_{m,K}(\Lambda)$ is an iso.

induction on m, $m=1$ is clear.

$x \longmapsto P([\pi])^m x$.

P min poly of π.

$P([\pi])$ ...

induction on m, $m=1$ is clear.

$$\begin{tikzcd} x \arrow[r, mapsto] & P([\pi])^m x \\ 0 \arrow[r] & W' \arrow[r] \arrow[d, "s"'] & \mathbb{B}_{m+1} \arrow[r] \arrow[d] & \mathbb{B}_m \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & W' \arrow[r] & \mathbb{B}_{m+1,K} \arrow[r] & \mathbb{B}_{m,K} \arrow[r] & 0 \\ P'(\pi)x & y \arrow[r, mapsto] & ([\pi]-\pi)^m x \end{tikzcd}$$

P min poly of π.

$\theta\left(\frac{P([\pi])}{[\pi]-\pi}\right) = P'(\pi) \neq 0$.

$P([\pi])$ is a gen of J_K.

$\square$.

$\mathbb{A}_{max,K} = \widehat{\mathbb{A}_{inf}[\frac{I_K}{\pi}]}_p$, $\mathbb{B}^+_{max,K} = \mathbb{A}_{max,K}$. ($K = K_0$, ignore K_0).

For $\Lambda \in \underline{Sym}$, $\mathbb{B}^+_{max,K}(\Lambda)$: equipped with a K-vectorspace norm by.

$\|x\|_{max} = 1$ iff $x \in \mathbb{A}_{max,K} - \pi\mathbb{A}_{max,K}$ (π-adic top).

I_K is gen by $[\pi]-\pi$ and π,

an element $x \in \mathbb{A}_{max,K}(\Lambda)$.

$$x = \sum_{n\geq 0} a_n \cdot \left(\frac{[\pi]-\pi}{\pi}\right)^n \quad \text{or} \quad \sum_{n\geq 0} b_n \cdot \left(\frac{[\pi]}{\pi}\right)^n$$

$a_n, b_n \in \mathbb{A}_{inf,K}(\Lambda) \longrightarrow 0$ $n \mapsto \infty$.

Ex: $K \otimes_{K_0} \mathbb{B}^+_{max} \xrightarrow{\sim} \mathbb{B}^+_{max,K}$.

Prop. 1) $\mathbb{B}^+_{max,K} \longrightarrow \mathbb{B}^+_{dR}$ is injective.

$$\sum a_n \left(\frac{[\pi]-\pi}{\pi}\right)^n \longmapsto \sum \left(\frac{a_n}{\pi}\right)^n \cdot ([\pi]-\pi)^n$$

2). $\mathbb{B}^+_{max,K}$ is a Banach Ring.

Pf. 1). For $v = \frac{[\pi]-\pi}{\pi}$ a generator of J_K.

$$\begin{tikzcd} \mathbb{A}_{inf,K}(\Lambda) \arrow[r] \arrow[d, hook] & \mathcal{O}_\Lambda[\pi X] \arrow[d, hook] \\ \mathbb{B}^+_{dR}(\Lambda) \arrow[r, "\tilde{\theta}_v"] & \Lambda[[X]] \\ \mathbb{A}_{max,K}(\Lambda) \arrow[u] \arrow[r, "\sim"] & \mathcal{O}_\Lambda\{X\} \arrow[u, hook] \end{tikzcd}$$

$$\begin{tikzcd} \omega \arrow[r, mapsto] \arrow[d] & \pi X \arrow[d, equal] \\ \pi v \arrow[r, mapsto] & \pi X \end{tikzcd}$$

$x \mapsto \sum_{n\geq 0} b_n(x) X^n$.

$\Rightarrow$ $\mathbb{A}_{max,K}(\Lambda) \xrightarrow{\sim} \mathcal{O}_\Lambda\{X\} = $ (p-adic completion of $\mathcal{O}_\Lambda[X]$.

2). $\mathbb{B}^+_{max,K}(\Lambda)$ is a p-adic Banach space.

T2). follows from that of $\mathbb{B}^+_{dR}$.

T1). Given $\lambda_0 \in \mathbb{B}^+_{max,K}(\Lambda)$, $s_0 \in Spec(\Lambda)$.

It suffices to show $\forall n$, $\exists$ $\ni s_0$ in $Spec(\Lambda)$ s.t. $s(\lambda) - s_0(\lambda_0) \in \pi^n A_{max,K}$. $\forall s \in U$.

$$\lambda_0 = \frac{1}{\pi^r} \sum_{k\geq 0} a_k \left(\frac{[\pi]}{\pi}\right)^k, \quad a_k \in \mathbb{A}_{inf,K}(\Lambda). \quad a_k \to 0 \ \pi\text{-adically}.$$

$$\lambda_0 = \frac{1}{\pi^r} \sum_{k\geq 0} a_k \left(\frac{[\pi]}{\pi}\right)^k, \quad a_k \in \mathbb{A}_{\inf,K}(\Lambda), \quad a_k \to 0 \ \pi\text{-adically}.$$

$\mathbb{A}_{\inf,K}$ is a Top Ring. (for the top. $\mathbb{I}_K^k(\Lambda) \subseteq \pi^k \mathbb{A}_{\max,K}(\Lambda)$).

$\Rightarrow$ above continuity.

$\Big($ More precise, $\exists k_0$ s.t. $a_k \in \pi^{n+r}$ $\forall k \geq k_0$,
$\forall k$. $\exists U_k \ni s_0$ s.t. $s(a_k) - s_0(a_k) \in \mathbb{I}_K^{n+r}(\Lambda) \subseteq \pi^{n+r} \mathbb{A}_{\max,K}(\Lambda)$.
$U = \bigcap_{k=1}^{k_0} U_k$, then $\forall s \in U$, $s(a_k (\frac{[\pi]}{\pi})^k) - s_0(-) \in \pi^{n+r} A_{\max,K} \Big)$. $\square$.

$\mathbb{B}_{st}^+ = \mathbb{B}_{\max}^+[u]$. $\circlearrowleft \varphi$, $N = -\frac{d}{du}$, $\mathbb{B}_{st,K}^+ = K \otimes_{K_0} \mathbb{B}_{st}^+$.

$$0 \to \mathbb{B}_{\max}^+ \to \mathbb{B}_{st}^+ \xrightarrow{N} \mathbb{B}_{st}^+ \to 0.$$

$\mathbb{B}_{\max,K}^+ \hookrightarrow \mathbb{B}_{dR}^+$ $\quad + \quad u \mapsto \log [p] = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \left(\frac{[p]}{p} - 1\right)^n$.

$\rightsquigarrow \quad \mathbb{B}_{st}^+ \to \mathbb{B}_{dR}^+$

Prop. It is injective. $\Big($ check it using $\widetilde{\Theta}_v$, $V = \frac{[p]-p}{p}$.
$\widetilde{\Theta}_v(\log [p]) = \log(1+X)$. $\Big)$.